Department of Mathematics

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Major Test (II Semester 2015 - 2016)

Time allowed: 2 hours **Max. Marks: 50**

1. Let X be a random variable with Poisson distribution with parameter λ. Show that the characteristic function of X is $\varphi_X(t) = exp\left[\lambda(e^{it} - 1)\right]$. Hence, compute $E(X^2)$, $Var(X)$ and $E(X^3)$.

   (2 + 1 + 1 + 1 marks)

2. Let X be a random variable with pdf

$$f(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2}, & 0 < x \leq 1 \\ \frac{1}{2x^2}, & 1 < x < \infty \end{cases}$$

   Find the pdf of the random variable $Y = \frac{1}{X}$. (4 marks)

3. Let X and Y be iid random variables each having uniform distribution on the interval $(\theta - 0.5, \theta + 0.5)$. Find the pdf of $X - Y$. (4 marks)

4. Let X, Y, Z be independent exponential distributed random variables with parameters λ, μ, ν respectively. Find $P(X < Y < Z)$.

   (4 marks)

5. (a) Define stochastic process.

   (b) Write two NOT stochastic processes from the following list: (i) Poisson process (ii) Wiener process (iii) Ornstein-Uhlenbeck process (iv) Levy process (v) Kolmogorov process (vi) Dirichlet process (vii) Markov process (viii) Erlang process (ix) Galton-Watson process (x) Cox process

   (2 + 2 marks)

6. Consider a DTMC with states 0, 1, 2, 3, 4. Suppose $p_{0,4} = 1$; and suppose that when the chain is in state i, $i > 0$, the next state is equally likely to be any of the states $0, 1, \ldots, i - 1$.

   (a) Discuss the nature of the states of this Markov chain. (3 marks)

   (b) Discuss whether there exist a limiting distribution and find one if it exists.

   (2 + 2 marks)

7. Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assume that successive plays of the game are independent. Suppose the gambler's fortune is presently i, and suppose that we know that the gambler's fortune will eventually reach N (before it goes to 0). Given this information, show that the probability he wins the next game is

$$\frac{p[1-(q/p)^{i+1}]}{1-(q/p)^{i}}, \quad \text{if } p \neq \frac{1}{2}$$

$$\frac{i+1}{2i}, \quad \text{if } p = \frac{1}{2}$$

   (5 marks)

irreducible → 1

recurrence, +ve recurrent → 1

~~aperiodic~~ ⇔

8. Consider the random telegraph signal, denoted by $X(t)$, jumps between two states, -1 and 1, according to the following rules. At time $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $P(X(0) = -1) = P(X(0) = 1) = 1/2$, and let the switching times be decided by a Poisson process $\{N(t), t \geq 0\}$ with parameter λ independent of $X(0)$. At time t, the signal

$$X(t) = X(0)(-1)^{N(t)}, t > 0.$$

Write the Kolmogorov forward equations for the continuous time Markov chain $\{X(t), t \geq 0\}$. Find the time-dependent probability distribution of $X(t)$ for any time t.

(2 + 3 marks)

9. Let $X(t)$ be the number of bacteria in a colony at time t. Evolution of the population is described by the time that each of the individuals takes for division in two, independently of the other bacteria and the life time of each bacterium also independent. Assume that, time for division is exponential distribution with rate λ and life time of each bacterium is also exponential distribution with rate μ. Without loss of generality, assume that $\{X(t), t \geq 0\}$ is modeled as a birth and death process.

   (a) Discuss the equilibrium probability distribution of the process.

   (b) Prove that, $\frac{dM(t)}{dt} = (\lambda - \mu)M(t)$ where $M(t)$ denotes the mean number of bacteria at time t.

   (c) Discuss $M(t)$ as $t \to \infty$.

(2 + 3 + 1 marks)

10. (a) Describe $M/M/1$ queuing model.

   (b) Derive the expression for the distribution of time spend in the system by any customer.

   (c) Deduce the mean waiting time from the above distribution.

(2 + 3 + 1 marks)

$r + 2\lambda r + \lambda$

$r(1+2\lambda) = -\lambda$

$r = -\frac{\lambda}{1+2\lambda}$

$1-2x = Ae^{-2\lambda t}$

$x = \frac{1}{2} - \frac{A}{2}e^{-2\lambda t}$

$t=0$

$\frac{1-A}{2} = \frac{0}{1}$

$\boxed{A=1}$

$\int \lambda e^{2\lambda t}\, dt$

$\Pi_{-1}(t) \cdot e^{2\lambda t}$

$\frac{\lambda}{2\lambda} e^{2\lambda t} + C$

$\frac{1}{2} e^{2\lambda t}$

$+C^{0}$