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Quiz 4
(15 min, Total: 15)

Computer communications networks (ELL785)

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1. [15] At all times, an urn contains N balls – some white balls and some black balls. At each stage, a coin having probability p, $0 < p < 1$, of landing heads is flipped. If heads appears, then a ball is chosen at random from the urn and is replaced by a white ball; if tails appears, then a ball is chosen from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the nth stage.
   (a) [2] Show **explicitly** whether X_n is a Markov process or not. If necessary, define the state.
   (b) [9] Compute the transition probabilities $P_{i,j}$: $P_{i,i}$, $P_{i,i+1}$ and $P_{i,i-1}$.
   (d) [2] Let $N = 2$. Find the proportion of time in each state.
   (e) [2] Based on your answer for $N = 2$, guess the answer for the limiting probability in the general case.

$P(\text{head}) = p$ $\qquad X_n \to$ no. of white balls.

(a) $P(X_n = k / X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \ldots X_0 = i_0)$

$= P(X_n = k / X_{n-1} = i_{n-1})$

If $X_{n-1} = i_{n-1}$, then ~~$X_n = i_{n-1}+1$ if head appears~~

if head appears, $X_n = \begin{cases} i_{n-1}, & \text{if ball picked is white} \\ i_{n-1}+1, & \text{if ball picked is black.} \end{cases}$

if tails appears, $X_n = \begin{cases} i_{n-1}-1, & \text{if ball picked is white} \\ i_{n-1}, & \text{if ball picked is black.} \end{cases}$

Clearly, it depends only on the fact what is the value in just previous state, and not any other state.

(b) $P(X_n = i / X_{n-1} = i) = p \times \frac{i}{N} + (1-p) \times \frac{N-i}{N}$

~~$P(X_n = i+1 / X_{n-1} = i) = \frac{1}{2} \times$~~

$= p - (1-p)\frac{i}{N} + (1-p)$

$= \frac{(2p-1)i}{N} + (1-p)$

$= P_{i,i}$

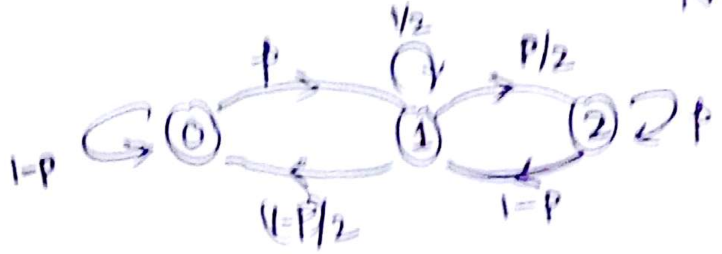
$\frac{2p-1}{2}$

$p - \frac{1}{2} + 1 - p$

$P_{i,i+1} = P(X_n = i+1 / X_{n-1} = i) = p \times \frac{N-i}{N}$ = P(head) × P(picking black ball from N balls)

$P_{i,i-1} = P(X_n = i-1 / X_{n-1} = i) = (1-p) \times \frac{i}{N}$ = P(tail) × P(picking white ball)

(d)



$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & & \left[\begin{matrix} 1-p & p & 0 \\ \frac{1-p}{2} & \frac{1}{2} & p/2 \\ 0 & 1-p & p \end{matrix}\right] & \\ 2 & & & \end{array}$$

$$\pi P = \pi$$

$$\begin{bmatrix} \Pi_0 & \Pi_1 & \Pi_2 \end{bmatrix} \begin{bmatrix} 1-p & p & 0 \\ \frac{1-p}{2} & \frac{1}{2} & p/2 \\ 0 & 1-p & p \end{bmatrix} = \begin{bmatrix} \Pi_0 & \Pi_1 & \Pi_2 \end{bmatrix}$$

$$\Pi_0(1-p) + \Pi_1\frac{(1-p)}{2} = \Pi_0$$

$$\Pi_0 p + \frac{\Pi_1}{2} + \Pi_2(1-p) = \Pi_1$$

$$\frac{\Pi_1 p}{2} + \Pi_2 p = \Pi_2 \Rightarrow \Pi_2(1-p) = \Pi_1 p$$

$$\Pi_2 = \frac{\Pi_1 p}{1-p}$$

$$\Pi_1 \frac{(1-p)}{2} = \Pi_0 p$$

$$\Pi_1 = \frac{2p\Pi_0}{1-p}$$

$$\Pi_0 p + \frac{p\Pi_0}{1-p} + 2\left(\frac{p}{1-p}\right)^2 \Pi_0 = 1$$

$$\Pi_0 \left[\frac{(1-p)^2 + p(1-p) + 2p^2}{(1-p)^2}\right] = 1$$

$$\Pi_0 \left[\frac{1 - 2p + \cancel{p^2} + p - \cancel{p^2} + 2p^2}{(1-p)^2}\right] = 1$$

$$\Pi_0 = \frac{(1-p)^2}{1-2p+2p^2}$$

$$\Pi_1 = \frac{2p\,\Pi_0}{1-p}$$

$$\Pi_2 = 2\left(\frac{p}{1-p}\right)^2 \Pi_0$$